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# Note

# How to decide the lark

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*Abstract*

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We present a decision algorithm for Smullyan's lark combinator. The method of choice consists in designing a canonical rewrite system which is confluent and terminating, or equivalently, Church–Rosser and strongly normalizing. This is related to the completion of an equational theory, but since traditional completion is divergent, we consider a set of rule *schemes* rather than a set of rules. This yields a polynomial decision procedure for the Lark combinator and many other fragments of combinatory logic.

## 1. Introduction

The aim of the present paper is to present an effective and efficient decision procedure for the word problem on the free term algebra $\mathbf{CL}(\mathbf{L})$, generated by Smullyan's [12] lark combinator $\mathbf{L}$ and a binary operation (application, denoted by juxtaposition), subject to the convertibility relation $=$ which is the smallest congruence extending the reduction relation induced by the rule $\mathbf{L}ab \rightarrow a(bb)$, for all $a, b \in \mathbf{CL}(\mathbf{L})$.

Recently Statman [13] proved that this combinator is decidable, but we feel that his procedure does not shed much light onto the possibilities for decision algorithms dealing with other fragments of combinatory logic. In contrast, our method for

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creating decision procedures can be applied to several other combinators as well. In fact, it is related to Knuth–Bendix completion (see [9]) based on the fact that the length of a term yields an (almost) well-founded ordering on classes of convertible terms for many combinators. However, in most cases the completion involves an infinite number of reduction rules.

For the following, we assume that the reader is familiar with combinatory reduction systems and has some background knowledge about reduction systems and combinatory logic in general. (Use, for instance, [6]; the standard references are [3, 5].)

## 2. First observations and informal survey

In combinatory logic based on standard combinators such as **S** and **K** or **B**, **C**, **I**, and **S** the convertibility relation (sometimes called *weak equality*) is not recursive – neither is any nontrivial set of terms which is closed under the convertibility relation.[1] Informally, the reason for this lies in the *combinatory completeness* of these systems, which allows representing all partial recursive functions. A natural question to ask might be: "What if we do not have combinatory completeness?". This leads to systems generated by combinators which are different from the ones mentioned above, and since one has to begin somewhere, one might even think of systems generated by one single combinator. But even then, bad news come in form of a proposition[2] stating that there are combinatorially complete systems generated by one single combinator (although examples of such basic combinators come with reduction rules lacking nice properties like being proper or regular). After this, one begins to wonder if there are any decidable subsystems of combinatory logic at all. As a matter of fact, there are many.

**Fact 2.1.** *Any reduction system with a strongly normalizing reduction relation satisfying the Church–Rosser property has a decidable convertibility relation.*

It is not difficult to verify, that all the combinators mentioned above are strongly normalizing, except **S**. In situations like these, the decision algorithm is very simple: To decide equality of two given terms, reduce them to their respective normal forms (which can be found in finite time because of strong normalization, and which are unique because of the Church–Rosser property), and compare the normal forms symbol by symbol. Questions about convertibility are, thus, translated into questions concerning the syntactical structure of normal forms, which are easy to answer.

Our method is derived from this observation, but we will not look for normal forms with respect to the natural reduction $\mathbf{L}ab \to a(bb)$. In the spirit of "short is simple", we will look for normal forms among the shortest possible representatives of a given class of convertible terms, and adapt the reduction relation accordingly. Because the

[1] See [6, Theorem 5.6] and [3, Theorem 6.6.2].
[2] [3, Proposition 8.1.4].

reduction $\rightarrow$ is not strongly normalizing, but the reverse reduction $\rightarrow^{-1}$ is, we choose the latter for our first try. Now, a new problem arises, because $\rightarrow^{-1}$ lacks the Church–Rosser property. Essentially, what we do is to define another notion of reduction, extending $\rightarrow^{-1}$ (which we call "jump reduction", since it boils down to jumping from any "backward" normal form to a canonical one). It will be Church–Rosser as well as strongly normalizing and, thus, will solve the decision problem.

## 3. The system CL(L)

Let $\mathbf{CL}(\mathsf{L})$ be the set of terms of combinatory logic built from the constant $\mathsf{L}$ alone by means of application. As usual, the abbreviation $a^n b$ is defined by $a^0 b \equiv b$ and $a^{n+1} b \equiv a(a^n b)$ for any $a, b \in \mathbf{CL}(\mathsf{L})$. As another short form, we use $\mathsf{L}_3 \equiv \mathsf{LLL}$.

**Definition 3.1.**

- On $\mathbf{CL}(\mathsf{L})$ the one-step reduction $\rightarrow$ is the smallest relation satisfying $\mathsf{L}ab \rightarrow a(bb)$ and $a \rightarrow b \Rightarrow (ac \rightarrow bc$ and $ca \rightarrow cb)$ for all $a, b, c \in \mathbf{CL}(\mathsf{L})$.
- The transitive, reflexive closure of $\rightarrow$ is denoted by $\twoheadrightarrow$ and the corresponding convertibility relation by $=$. (Note that $=$ is the symmetric closure of $\twoheadrightarrow$.)

We need a few more general notions and a basic result.

**Definition 3.2.** Let $X$ be a set and $\rightarrow \subseteq X \times X$ a relation on $X$. Again denote the transitive, reflexive closure of $\rightarrow$ by $\twoheadrightarrow$.

- A (finite or infinite) sequence $a_1, a_2, \ldots$ in $X$ is called $\rightarrow$*-chain*, if $a_i \rightarrow a_{i+1}$, for all $i$.
- $\rightarrow$ is called *strongly normalizing* (SN) or *terminating*, iff there are no infinite $\rightarrow$-chains in $X$.
- is called *weakly Church–Rosser* (WCR) or *locally confluent*, iff for all $a, b, c \in X$ with $a \rightarrow b$ and $a \rightarrow c$, there exists $d \in X$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.
- $\rightarrow$ is called *Church–Rosser* (CR) or *confluent*, iff for any $a, b, c \in X$ with $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$, there exists $d \in X$ such that $b \twoheadrightarrow d$ and $c \twoheadrightarrow d$.

**Proposition 3.3.** *A relation which is strongly normalizing and satisfies the weak Church–Rosser property satisfies the Church–Rosser property* ($SN + WCR \Rightarrow CR$).

**Proof.** See [11]. □

Whenever a reduction relation $\rightarrow$ is both SN and CR, there is a simple solution for the word problem of the corresponding congruence $=$: Just find the normal forms[3] of two given words (whose existence and uniqueness are granted by SN and CR) and compare them: The original two words are equal in the sense of the congruence $=$, iff

[3] Recall that a term $c$ is in *normal form* if there is no $d$ such that $c \rightarrow d$ or, equivalently, if $c$ contains no redex as subterm.

their respective normal forms are identical. But, as most of the natural reductions in combinatory logic are not normalizing at all, there are usually only partial methods, e.g., the so-called *leftmost-reduction strategy* (for more information, see [3]), which does reduce a term to normal form *whenever it has one*. We also find this situation in the case of $\mathbf{CL}(\mathsf{L})$, where we have infinitely many terms without normal forms, e.g., $\mathsf{LL}(\mathsf{LL})$:

$$\mathsf{LL}(\mathsf{LL})\to\mathsf{L}(\mathsf{LL}(\mathsf{LL}))\to\cdots\to\mathsf{L}^n(\mathsf{LL}(\mathsf{LL}))\to\cdots$$

But at least we have the following theorem.

**Theorem 3.4.** *The reduction $\to$ satisfies the Church–Rosser property.*

**Proof.** See [7] or modify one of the standard proofs for CR in the lambda calculus or combinatory logic. $\square$

We saw that we will not succeed in assigning a canonical representative to every class of convertible terms, if we follow the natural reduction $\to$ on $\mathbf{CL}(\mathsf{L})$, leading from $\mathsf{L}ab$ to $a(bb)$ – the terms are much more likely to grow to unlimited length. This leads to the idea that there may be some way to track down, not, where we are going with reduction, but where we come from. If one compares a $\to$-redex $\mathsf{L}ab$ with its contractum $a(bb)$, one gets the strong feeling that the reverse reduction would be SN.

But what about CR, which holds for $\to$? As it was shown by G. Plotkin (cited in [3, Exercise 3.5.11(vii)]), one cannot conclude the "upside-down"-CR property for a relation from its being CR. Indeed, there are counter-examples for the backward $\to$-reduction in $\mathbf{CL}(\mathsf{L})$:

$$\mathsf{L}(\mathsf{L}a)b\twoheadrightarrow a(bb(bb)) \quad\text{and}\quad a(\mathsf{LL}bb)\twoheadrightarrow a(bb(bb)),$$

but there is no common reduct, i.e., $c\in\mathbf{CL}(\mathsf{L})$ such that

$$c\twoheadrightarrow\mathsf{L}(\mathsf{L}a)b \quad\text{and}\quad c\twoheadrightarrow a(\mathsf{LL}bb).$$

Hence, we cannot use directly the inverse $\to^{-1}$ of $\to$, since the above example shows that $\to^{-1}$ violates CR, which is intolerable for decision purposes. We will, hence, have to introduce a new notion of reduction (based on $\to^{-1}$). Recall for the following definition that $\mathsf{L}_3\equiv\mathsf{LLL}$.

**Definition 3.5.** On $\mathbf{CL}(\mathsf{L})$ we define the reduction $\mapsto$ ("jump") as follows:

(R1) $a(bb)\mapsto_{\mathrm{R}1}\mathsf{L}ab$.

(R2) $a(\mathsf{LL}bb)\mapsto_{\mathrm{R}2}\mathsf{L}(\mathsf{L}a)b$.

(R3) $a(\mathsf{L}^{n+1}(\mathsf{L}^n\mathsf{L}_3b)b)\mapsto_{\mathrm{R}3}\mathsf{L}^{n+3}ab$ for $n\geqslant 0$.

(R4) $a(\mathsf{L}_3\mathsf{LL})\mapsto_{\mathrm{R}4}\mathsf{L}^3a\mathsf{L}$.

(R5) $a(\mathsf{L}^n(\mathsf{LL}(\mathsf{L}^n\mathsf{L}_3))(\mathsf{L}^n\mathsf{L}_3))\mapsto_{\mathrm{R}5}\mathsf{L}^{n+3}a(\mathsf{L}^n\mathsf{L}_3)$ for $n\geqslant 1$.

(R6) $\mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3 \mapsto_{R6} \mathbf{L}(\mathbf{L}a)\mathbf{L}_3$.

(R7) $\mathbf{L}^2(\mathbf{LL}a)\mathbf{L}_3 \mapsto_{R7} \mathbf{L}(\mathbf{LL}a)\mathbf{L}_3$.

(R8) $\mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3 \mapsto_{R8} \mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3$ for $n \geqslant 0$.

(R9) $\mathbf{LL}_3\mathbf{L}_3 \mapsto_{R9} \mathbf{L}_3\mathbf{L}_3$.

(R10) $\mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 \mapsto_{R10} \mathbf{LL}(\mathbf{L}^n\mathbf{L}_3)\mathbf{L}_3$ for $n \geqslant 0$.

Additionally, we require that $a \mapsto_{Ri} b \Rightarrow ac \mapsto_{Ri} bc$ and $ca \mapsto_{Ri} cb$ for arbitrary $a, b, c \in \mathbf{CL}(\mathbf{L})$, and any of the above reduction rules (R$i$). Finally, $\mapsto$ is given by $a \mapsto b$ iff $a \mapsto_{Ri} b$ by one of the rules (R$i$).

As usual, we denote the transitive, reflexive closure of $\mapsto$ by $\mapsto\!\!\!\!\to$.

What is the idea behind $\mapsto$? Obviously, one has $a \to b \Rightarrow b \mapsto a$, which guarantees that $\mapsto$ generates a congruence at least as big as $\to$ does. (Later on we will show that they are the same.) This slightly exotic relation is best regarded as the limit of a process which, starting from the inverse of $\to$, fixes the CR property by introducing new kinds of rules, thus relating formerly unrelated normal forms. Every step (i.e., every $\mapsto_{Ri}$) generates some critical branches in the backward reduction which may not be confluent and, hence, violate CR (even WCR). Jump reduction ensures, that, at every level of this process, we can "jump" from one branch to another. As we shall prove, we will not jump forever.

**Lemma 3.6.** *The reduction* $\mapsto$ *is strongly normalizing.*

**Proof.** Define $|\ |: \mathbf{CL}(\mathbf{L}) \to \mathbb{N}$ (the length of a term) inductively by $|\mathbf{L}| := 1$ and $|(ab)| := |a| + |b|$. Because of the compatibility of $|\ |$ with $\mathbf{CL}(\mathbf{L})$'s term building process, one only has to show that the length of a $\mapsto$-redex is never less than the length of its contractum. A simple calculation shows that the length $|\ |$ is strictly decreased by any contraction step, except in the case of $a(\mathbf{LL}) \mapsto \mathbf{L}a\mathbf{L}$, where it remains unchanged. Towards a contradiction, assume that there is an infinite $\mapsto$-reduction sequence $a_0 \mapsto a_1 \mapsto \cdots$ It follows that there is a number $n$ such that for any $i \geqslant n$, $a_i \mapsto a_{i+1}$ by contraction of a redex-occurrence of the form $a(\mathbf{LL})$. Moreover, since there are only finitely many terms of a given length, our infinite reduction sequence must become circular and "loops" over a finite set of terms.

Furthermore, if a contraction $a(\mathbf{LL}) \mapsto \mathbf{L}a\mathbf{L}$ does not decrease the number of redex-occurrences of the form $x(\mathbf{LL})$ in the term containing $a\mathbf{LL}$, then $a \equiv \mathbf{LL}$: $\mathbf{L}a \equiv \mathbf{L}(\mathbf{LL})$ is then again a redex of this form, and the number of redex-occurrences of this kind remains unchanged during this contraction. But even if all the $\mapsto$-redexes in a term are of this particular form, all of them will be $\mapsto$-contracted after finitely many steps, since the contraction of a *contractum* of such a redex *does* decrease the number of these particular redex-occurrences by one. So, the number of length-preserving redex-occurrences *does* decrease sooner or later; hence, $\mapsto$-reduction cannot loop, and infinite $\mapsto$-reductions sequences are, thus, impossible. $\square$

**Proposition 3.7.** *The reduction relation $\mapsto$ satisfies the weak Church–Rosser property.*

**Proof.** In order to improve readability, we postpone this technical and lengthy proof to the last section. $\square$

According to Lemma 3.6 and Propositions 3.3 and 3.7, we have shown that $\mapsto$ is CR, which is essential for the proof of the following proposition.

**Proposition 3.8.** *Let $a, b \in \mathbf{CL}(\mathbf{L})$. Then $a = b$, iff there is $c \in \mathbf{CL}(\mathbf{L})$ such that $a \mapsto\!\!\!\twoheadrightarrow c$ and $b \mapsto\!\!\!\twoheadrightarrow c$.*

**Proof.** For the proof of the implication $\Leftarrow$ it is enough to show that $\mapsto$-reduction preserves equality, the convertibility relation generated by $\to$. We have to verify that all the redexes of all the rules in Definition 3.5 are equal to their contracta:

(R1) $a(bb) = \mathbf{L}ab$ by definition.

(R2) $a(\mathbf{LL}bb) = \mathbf{L}(\mathbf{L}a)b$ because of the reductions $\mathbf{L}(\mathbf{L}a)b \twoheadrightarrow a(bb(bb)) \twoheadleftarrow a(\mathbf{LL}bb)$.

(R3) $a(\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b) = \mathbf{L}^{n+3}ab$. This needs induction on the exponent $n$. If $n=0$, the statement is true by $a(\mathbf{L}(\mathbf{L}_3 b)b) \twoheadrightarrow a(bb(bb)(bb(bb))) \twoheadleftarrow \mathbf{L}^3 ab$. For $n>0$, we can use the derivation $a(\mathbf{L}^{n+2}(\mathbf{L}^{n+1}\mathbf{L}_3 b)b) \twoheadrightarrow a(\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3(bb))(bb)) = \mathbf{L}^{n+3}a(bb)$ by the induction hypothesis, but also $\mathbf{L}^{n+4}ab \to \mathbf{L}^{n+3}a(bb)$.

(R4) $a(\mathbf{L}_3\mathbf{LL}) = \mathbf{L}^3 a\mathbf{L}$. In this case we have $a(\mathbf{L}_3\mathbf{LL}) \twoheadrightarrow a(\mathbf{L}(\mathbf{LL}(\mathbf{LL}))\mathbf{L}) \leftarrow a(\mathbf{L}(\mathbf{L}_3\mathbf{L})\mathbf{L}) = \mathbf{L}^3 a\mathbf{L}$, where the last step comes from the already justified equation for (R3).

(R5) $a(\mathbf{L}^n(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3)) = \mathbf{L}^{n+3}a(\mathbf{L}^n\mathbf{L}_3)$. This follows from $a(\mathbf{L}^n(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3)) \to a(\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3(\mathbf{L}^n\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3)) = \mathbf{L}^{n+3}a(\mathbf{L}^n\mathbf{L}_3)$ by the equation for (R3) above.

(R6) $\mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3 = \mathbf{L}(\mathbf{L}a)\mathbf{L}_3$ is a consequence of $\mathbf{L}^3 a\mathbf{L}_3 = a(\mathbf{L}(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \leftarrow a(\mathbf{LL}\mathbf{L}_3\mathbf{L}_3) = \mathbf{L}^2 a\mathbf{L}_3$ by (R3) and (R2).

(R7) $\mathbf{L}^2(\mathbf{LL}a)\mathbf{L}_3 = \mathbf{L}(\mathbf{LL}a)\mathbf{L}_3$ is implied by $\mathbf{L}^2(\mathbf{LL}a)\mathbf{L}_3 \to \mathbf{L}^2(\mathbf{L}(aa))\mathbf{L}_3 = \mathbf{L}(\mathbf{L}(aa))\mathbf{L}_3 \leftarrow \mathbf{L}(\mathbf{LL}a)\mathbf{L}_3$, where the equation in the middle is proved in (R6).

(R8) $\mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3 = \mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3$. This is proved by induction on $n$. If $n=0$, then $\mathbf{L}^2(\mathbf{L}_3 a)\mathbf{L}_3 \twoheadrightarrow \mathbf{L}^2(\mathbf{LL}(aa))\mathbf{L}_3 = \mathbf{L}(\mathbf{LL}(aa))\mathbf{L}_3 \twoheadleftarrow \mathbf{L}(\mathbf{L}_3 a)\mathbf{L}_3$, using (R7). For $n>0$, consider $\mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3 \to \mathbf{L}^2(\mathbf{L}^{n-1}\mathbf{L}_3(aa))\mathbf{L}_3 = \mathbf{L}(\mathbf{L}^{n-1}\mathbf{L}_3(aa))\mathbf{L}_3 \leftarrow \mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3$.

(R9) $\mathbf{LL}_3\mathbf{L}_3 = \mathbf{L}_3\mathbf{L}_3$. This comes from $\mathbf{LL}_3\mathbf{L}_3 \to \mathbf{L}^2(\mathbf{LL})\mathbf{L}_3 = \mathbf{L}(\mathbf{LL})\mathbf{L}_3 \leftarrow \mathbf{L}_3\mathbf{L}_3$, using the result of (R6).

(R10) $\mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 = \mathbf{LL}(\mathbf{L}^n\mathbf{L}_3)\mathbf{L}_3$. This follows by application or (R6) in $\mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 \to \mathbf{L}(\mathbf{L}(\mathbf{L}^n\mathbf{L}_3(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 = \mathbf{L}(\mathbf{L}^n\mathbf{L}_3(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 \leftarrow \mathbf{LL}(\mathbf{L}^n\mathbf{L}_3)\mathbf{L}_3$.

For the other implication $\Rightarrow$ suppose that $a = b$. Then, since $\to$ is CR, there is $c \in \mathbf{CL}(\mathbf{L})$ with $a \twoheadrightarrow c$ and $b \twoheadrightarrow c$ and, thus, $c \mapsto\!\!\!\twoheadrightarrow a$ and $c \mapsto\!\!\!\twoheadrightarrow b$, since $\to^{-1}$ is contained in $\mapsto$. But $\mapsto$ is also CR; so, we get the existence of $d \in \mathbf{CL}(\mathbf{L})$ with $a \mapsto\!\!\!\twoheadrightarrow d$ and $b \mapsto\!\!\!\twoheadrightarrow d$. $\square$

Now the decidability of equality in $\mathbf{CL}(\mathbf{L})$ follows easily with the following theorem.

**Theorem 3.9.** *For $a, b \in \mathbf{CL}(\mathbf{L})$ one has $a = b$, iff a and b have the same* $\mapsto\!\!\!\rightarrow$*-normal form.*

**Proof.** Immediate from Lemma 3.6 and Propositions 3.7 and 3.8. $\square$

In order to decide whether two terms $a, b \in \mathbf{CL}(\mathbf{L})$ are equal ($\rightarrow$-convertible), simply $\mapsto\!\!\!\rightarrow$-reduce them to $\mapsto\!\!\!\rightarrow$-normal form, which is a process that always terminates by strong normalization, and then check whether the resulting terms are identical or not.

## 4. Conclusions

In this (logically) last section we would like to make some remarks about how this result relates to other results in the field.

First, note that the complexity of this decision procedure is a polynomial of low degree, since it is based on rewriting, whereas Statman's [13] procedure does not seem to easily allow complexity estimates. Additionally, our procedure lends itself to an extended system including not only ground terms, but also variables, without modification.

Furthermore, the decision algorithm based on the jump reduction works not only for $\mathbf{CL}(\mathbf{L})$. We have been able to apply this method and carry out analogous decidability proofs to other combinators, e.g., $\mathbf{W}$ and $\mathbf{M}_1$ with the following reduction rules:

$$\mathbf{W}ab \rightarrow abb,$$

$$\mathbf{M}_1 ab \rightarrow aab.$$

It is worth noting that $\mathbf{W}$ is one of Curry's basic combinators,[4] and that $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{I}$ all have strongly normalizing reductions and, hence, generate decidable structures as long as we consider them one at a time. However, the system generated by the basis $\{\mathbf{B}, \mathbf{C}, \mathbf{I}, \mathbf{W}\}$ is known to be of the same expressive power as the $\lambda\mathbf{I}$-calculus;[5] hence, conversion in this system is not decidable. So, we conclude that decidability may get lost, when we proceed from "good" systems to their union (i.e., the system generated by the union of the bases of the well-behaved systems), equipped with the least compatible conversion containing all the basic conversions. It is, thus, natural to ask: Under what exact conditions will decidability disappear? An answer to this question might also be of some help in finding other decision methods. This kind of solution to the decision problem gives also some explicit knowledge about the structure of the system under consideration, here $\mathbf{CL}(\mathbf{L})$. (Would you have guessed before, that $\mathbf{L}^k \mathbf{L}_3 \mathbf{L}_3 = \mathbf{L}_3 \mathbf{L}_3$ for any $k$?)

As we mentioned earlier, this method is related to Knuth–Bendix completion (also called critical-pair completion). In principle, it would be possible to substitute known

[4] See [5].

[5] See [3, Proposition 9.3.5, and the proof of Proposition 9.3.7] for a definition of $\mathbf{S}$ using $\mathbf{C}$, $\mathbf{B}$ and $\mathbf{W}$.

results about infinite completions for our proof. (See [1, 2, 4, 8, 10]). However, in order to apply these results, all the explicit knowledge, we produce for our proof, would be necessary.

## 5. Proof of Proposition 3.7

This section is devoted to the proof of Proposition 3.7 which is crucial to this approach to decidability.

**Proposition 3.7.** *The reduction relation* $\mapsto$ *satisfies the weak Church Rosser property.*

**Proof.** The plan of the proof is as follows: All situations where the CR property might possibly be violated, have to be investigated. This can only happen when there is a term allowing simultaneously two different reductions to be applied. Our task will be to verify that all those pairs of co-initial reductions are confluent. Consider two co-initial $\mapsto$-reductions, induced by redexes $\Delta_1$ and $\Delta_2$, respectively:

$$B_1 \overset{\Delta_1}{\leftarrow\!\shortmid} B \overset{\Delta_2}{\mapsto} B_2.$$

First, we dispose of the trivial cases: If $\Delta_1$ and $\Delta_2$ are nonoverlapping, it is easy to find a common reduct by just reducing one after the other. The same is true for $\Delta_1 \equiv \Delta_2$ since all of the redexes of the reduction rules are $\mapsto$-"irreducible" with respect to the other rules, i.e., the very same rule has to be applied to both $\Delta_1$ and $\Delta_2$, thus yielding identical results.

In all other cases, we may assume that $\Delta_2$ is a subterm occurrence inside $\Delta_1$. The possible forms of $\Delta_1$ and $\Delta_2$ are given in Definition 3.5 and also listed below.

Next we mention that all the cases, where $\Delta_2$ occurs in a variable subterm of $\Delta_1$ are straightforward: Since, if $a \overset{\Delta_2}{\mapsto} a'$ or $b \overset{\Delta_2}{\mapsto} b'$, then we have, for instance,

$$a(\mathbf{L}^{n+1}(\mathbf{L}^n(\mathbf{LLL})b)b) \mapsto\!\!\!\to a'(\mathbf{L}^{n+1}(\mathbf{L}^n(\mathbf{LLL})b')b') \mapsto \mathbf{L}^{n+3}a'b'$$

as well as

$$a(\mathbf{L}^{n+1}(\mathbf{L}^n(\mathbf{LLL})b)b) \mapsto \mathbf{L}^{n+3}ab \mapsto\!\!\!\to \mathbf{L}^{n+3}a'b'$$

and likewise for the redexes of all other types.

For the nontrivial collisions, it is enough to focus on the nonvariable subterms inside the redexes containing a (meta-)variable, excluding the whole term, which are

- $bb$ in $a(bb)$,
- $\mathbf{LL}b$ and $\mathbf{LL}bb$ in $a(\mathbf{LL}bb)$,
- $\mathbf{L}^n\mathbf{L}_3 b$, $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3 b)$, $\ldots$, $\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)$, $\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b$ in $a(\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b)$,
- None in $a(\mathbf{L}_3\mathbf{LL})$,

- None in $a(\mathbf{L}^n(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3))$,
- $\mathbf{L}a$ and $\mathbf{L}(\mathbf{L}a)$ in $\mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3$,
- $\mathbf{LL}a$ and $\mathbf{L}(\mathbf{LL}a)$ in $\mathbf{L}^2(\mathbf{LL}a)\mathbf{L}_3$,
- $\mathbf{L}^n\mathbf{L}_3a$ and $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3a)$ in $\mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3a)\mathbf{L}_3$,
- None in $\mathbf{L}\mathbf{L}_3\mathbf{L}_3$,
- None in $\mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3$.

In the following main part of the proof, we will consider all these redexes of any type and all their critical subterm occurrences. To every co-initial pair of contractions we give two reductions leading to a common reduct.

$\Delta_1 \equiv a(bb)$: The only critical subterm is $bb$ which we have to match against the redexes of all the reduction rules (R1)–(R10):

- $\Delta_2 \equiv bb \equiv x(yy), x(\mathbf{LL}yy), x(\mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3y)y), x(\mathbf{L}_3\mathbf{LL})$, or $x(\mathbf{L}^n(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3))$:
  We can treat those five rules in common since we do not need knowledge about their exact form. Note that rules (R1)–(R5) are all of the form $a\Gamma[b] \mapsto_r \mathbf{L}^\gamma a\Theta[b]$, where $\Gamma[b]$ and $\Theta[b]$ are the appropriate terms which may [in (R1)–(R3)] or may not [in (R4) and (R5)] depend on $b$.
  From $bb \equiv x\Gamma[y]$, we have $b \equiv x \equiv \Gamma[y]$. The co-initial contractions and continuations to a common reduct are

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R1}} \mathbf{L}a\Gamma[y] \mapsto_r \mathbf{L}^{\gamma+1}ay,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_r a(\mathbf{L}^\gamma\Gamma[y]\Theta[y]) \equiv a(\mathbf{L}^{\gamma-1}(\mathbf{L}\Gamma[y])\Theta[y])$$

$$\mapsto_r a(\mathbf{L}^{\gamma-1}(\mathbf{L}^\gamma\mathbf{L}\Theta[y])\Theta[y]) \mapsto_{\mathrm{R1}} a(\mathbf{L}^{\gamma-1}(\mathbf{L}^{\gamma-2}\mathbf{L}_3\Theta[y])\Theta[y])$$

$$\mapsto_{\mathrm{R3}} \mathbf{L}^{\gamma+1}a\Theta[y].$$

  There is a slight modification for $\gamma=1$: This would cause $\mathbf{L}^{\gamma-2}\mathbf{L}_3$ to be undefined. In this case, use the reduction sequence

$$\Delta_1 \mapsto\!\!\!\!\rightarrow a(\mathbf{L}^{\gamma-1}(\mathbf{L}^\gamma\mathbf{L}\Theta[y])\Theta[y]) \equiv a(\mathbf{LL}\Theta[y]\Theta[y]) \mapsto_{\mathrm{R2}} \mathbf{L}^{\gamma+1}a\Theta[y].$$

  We will refer to this scheme as *generic reduction* for the rules (R1)–(R5). Whenever $\gamma-2$ appears in a generic reduction later on, the reader is invited to account for this by himself.
- $\Delta_2 \equiv bb \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3x)\mathbf{L}_3, \mathbf{L}\mathbf{L}_3\mathbf{L}_3$, or $\mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3$. This is because the right subterm is everywhere $\mathbf{L}_3$; so, we would have $b \equiv \mathbf{L}_3$ and $\Delta_2 \equiv bb \equiv \mathbf{L}_3\mathbf{L}_3$, which is not true for any of the listed. So, there are no other nontrivial co-initial contractions.

$\Delta_1 \equiv a(\mathbf{LL}bb)$: There are the two critical subterms $\mathbf{LL}b$ and $\mathbf{LL}bb$.

- $\Delta_2 \equiv \mathbf{LL}b$: For the generic reduction, we have $\mathbf{LL}b \equiv x\Gamma[y]$; thus, $x \equiv \mathbf{LL}$ and $b \equiv \Gamma[y]$. The reductions are

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R2} \mathbf{L}^2 a\Gamma[y] \mapsto_\Gamma \mathbf{L}^{\gamma+2} a\Theta[y],$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_\Gamma a(\mathbf{L}^\gamma(\mathbf{LL})\Theta[y]\Gamma[y]) \mapsto_\Gamma a(\mathbf{L}^\gamma(\mathbf{L}^\gamma(\mathbf{LL})\Theta[y])\Theta[y])$$

$$\mapsto_{R1} a(\mathbf{L}^\gamma(\mathbf{L}^{\gamma-1}\mathbf{L}_3\Theta[y])\Theta[y]) \mapsto_{R3} \mathbf{L}^{\gamma+2} a\Theta[y].$$

- $\Delta_2 \equiv \mathbf{LL}b \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 x)\mathbf{L}_3, \mathbf{LL}_3\mathbf{L}_3, \mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3$. The argument is here that the left subterm of none of them is $\mathbf{LL}$.
- $\Delta_2 \equiv \mathbf{LL}bb \equiv x\Gamma[y]$; thus, $x \equiv \mathbf{LL}b \equiv \mathbf{LL}\Gamma[y]$. This is an analogue to the generic reductions for $\Delta_2 \equiv \mathbf{LL}b$ above. The only change is in

$$\Delta_1 \overset{\Delta_2}{\mapsto}_\Gamma a(\mathbf{L}^\gamma(\mathbf{LL})\Gamma[y]\Theta[y]) \mapsto_\Gamma a(\mathbf{L}^\gamma(\mathbf{L}^\gamma(\mathbf{LL})\Theta[y])\Theta[y]).$$

- $\Delta_2 \equiv \mathbf{LL}bb \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 x)\mathbf{L}_3$. The left subterm should be $\mathbf{LLL}_3$ which is never the same as $\mathbf{L}(\mathbf{L}(\ldots))$ regardless of what $x$ we take.
- $\Delta_2 \equiv \mathbf{LL}bb \not\equiv \mathbf{LL}_3\mathbf{L}_3, \mathbf{L}(\mathbf{LL}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3$.

$\Delta_1 \equiv a(\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 bb))$: Now we have many critical subterms. For $\mathbf{L}^n\mathbf{L}_3 b$ and $\mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b$ we can use generic reductions, for the others of form $\mathbf{L}^k(\mathbf{L}^n\mathbf{L}_3 b)$ we have to be more specific.

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 b \equiv x\Gamma[y]$; thus, $x \equiv \mathbf{L}^n\mathbf{L}_3$ and $b \equiv \Gamma[y]$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^{n+3} ab \mapsto_\Gamma \mathbf{L}^{n+\gamma} a\Theta[y],$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_\Gamma a(\mathbf{L}^{n+1}(\mathbf{L}^{n+\gamma}\mathbf{L}_3\Theta[y])b)$$

$$\mapsto\!\!\!\!\mapsto_\Gamma a(\mathbf{L}^{n+1+\gamma}(\mathbf{L}^{n+\gamma}\mathbf{L}_3\Theta[y])\Theta[y]) \mapsto_{R3} \mathbf{L}^{n+\gamma+3} a\Theta[y].$$

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 b \equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3$: This implies that $b \equiv \mathbf{L}_3$, as well as $\mathbf{L}^n\mathbf{L}_3 \equiv \mathbf{L}^2(\mathbf{L}x)$. Writing this as $\mathbf{L}^2(\mathbf{L}(\mathbf{L}^{n-3}x)) \equiv \mathbf{L}^2(\mathbf{L}x)$ we see that this works only for $n \geqslant 2$ and $x \equiv \mathbf{L}_3$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^{n+3} a\mathbf{L}_3 \mapsto_{R6} \mathbf{L}^{n+2} a\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{R6} a(\mathbf{L}^{n+1}(\mathbf{L}^{n-1}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R7} a(\mathbf{L}^n(\mathbf{L}^{n-1}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3)$$

$$\mapsto_{R3} \mathbf{L}^{n+2} a\mathbf{L}_3.$$

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 b \equiv \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3$. This can only be satisfied by setting $b \equiv \mathbf{L}_3$, $x \equiv \mathbf{L}$, and $n = 2$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^5 a\mathbf{L}_3 \mapsto_{R6} \mathbf{L}^4 a\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{R7} a(\mathbf{L}^3(\mathbf{LL}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R8} a(\mathbf{L}^2(\mathbf{LL}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R3} \mathbf{L}^4 a\mathbf{L}_3.$$

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 b \not\equiv \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3$, because we should then have $\mathbf{L}^{n-2}\mathbf{L}_3 \equiv \mathbf{L}^m\mathbf{L}_3 x$ which is not possible.
- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}^3 b \equiv \mathbf{L}\mathbf{L}_3\mathbf{L}_3$. Thus, $b \equiv \mathbf{L}_3$ and $n=1$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^4 a\mathbf{L}_3 \mapsto_{R6} \mathbf{L}^3 a\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{R9} a(\mathbf{L}^2(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R8} a(\mathbf{L}(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R3} \mathbf{L}^3 a\mathbf{L}_3.$$

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}^3 b \not\equiv \mathbf{L}(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$.
- $\Delta_2 \equiv \mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b \equiv x\Gamma[y]$: There is only one minor change compared to the generic reductions of $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 b$:

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\Gamma} a(\mathbf{L}^{n+\gamma+1}(\mathbf{L}^n\mathbf{L}_3 b)\Theta[y])$$

$$\mapsto_{\Gamma} a(\mathbf{L}^{n+1+\gamma}(\mathbf{L}^{n+\gamma}\mathbf{L}_3\Theta[y])\Theta[y]) \twoheadrightarrow \mathbf{L}^{n+\gamma+3} a\Theta[y].$$

- $\Delta_2 \equiv \mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b \not\equiv \mathbf{L}^2(\mathbf{L}\mathbf{L}x)\mathbf{L}_3$, because $\mathbf{L}^{n-1}(\mathbf{L}^n\mathbf{L}_3\mathbf{L}_3) \equiv \mathbf{L}\mathbf{L}x$ is impossible.
- $\Delta_2 \equiv \mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b \equiv \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3$ implies $b \equiv x \equiv \mathbf{L}_3$ and $n=m=1$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^4 a\mathbf{L}_3 \mapsto_{R6} \mathbf{L}^3 a\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R7} a(\mathbf{L}(\mathbf{L}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R9} a(\mathbf{L}(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3) \mapsto_{R3} \mathbf{L}^3 a\mathbf{L}_3.$$

- $\Delta_2 \equiv \mathbf{L}^{n+1}(\mathbf{L}^n\mathbf{L}_3 b)b \not\equiv \mathbf{L}\mathbf{L}_3\mathbf{L}_3$ or $\mathbf{L}(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))(\mathbf{L}^m\mathbf{L}_3)$.
- $\Delta_2 \equiv \mathbf{L}^k(\mathbf{L}^n\mathbf{L}_3 b) \equiv x(yy)$: This is only possible when $k=1$ and $b \equiv y \equiv \mathbf{L}^n\mathbf{L}_3$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^{n+3} a(\mathbf{L}^n\mathbf{L}_3),$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{R1} a(\mathbf{L}^n(\mathbf{L}\mathbf{L}(\mathbf{L}^n\mathbf{L}_3))(\mathbf{L}^n\mathbf{L}_3)) \mapsto_{R5} \mathbf{L}^{n+3} a(\mathbf{L}^n\mathbf{L}_3).$$

- $\Delta_2 \equiv \mathbf{L}^k(\mathbf{L}^n\mathbf{L}_3 b) \equiv x(\mathbf{L}\mathbf{L}yy)$: This cannot happen unless $k=1$, $n=0$, and $b \equiv y \equiv \mathbf{L}$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{R3} \mathbf{L}^3 a\mathbf{L},$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{R1} a(\mathbf{L}^2\mathbf{L}\mathbf{L}) \mapsto_{R4} \mathbf{L}^3 a\mathbf{L}.$$

- $\Delta_2 \equiv \mathbf{L}^k(\mathbf{L}^n\mathbf{L}_3 b)$ cannot be equivalent to $x(\mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y), x(\mathbf{L}_3\mathbf{L}\mathbf{L})$, or $x(\mathbf{L}^m(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))(\mathbf{L}^m\mathbf{L}_3))$, because $\mathbf{L}^{k-1}(\mathbf{L}^n\mathbf{L}_3 b) \not\equiv \mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y, \mathbf{L}_3\mathbf{L}\mathbf{L}$, or $\mathbf{L}^m(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))(\mathbf{L}^m\mathbf{L}_3)$.

- $\mathbf{L}^k(\mathbf{L}^n\mathbf{L}_3 b)$ does not match $\mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3, \mathbf{LL}_3\mathbf{L}_3$, or $\mathbf{L}(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$: $\Delta_2$ has the form $\mathbf{L}(\ldots)$, whereas the term on the right-hand side display the structure $\mathbf{L}(\ldots)\mathbf{L}_3$.

$\Delta_1 \equiv \mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3$: There are $\mathbf{L}a$ and $\mathbf{L}(\mathbf{L}a)$ as critical subterms.
- $\Delta_2 \equiv \mathbf{L}a \equiv x\Gamma[y]$; therefore, $x \equiv \mathbf{L}$ and $a \equiv \Gamma[y]$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R6}} \mathbf{L}(\mathbf{L}\Gamma[y])\mathbf{L}_3 \mapsto_\Gamma \mathbf{L}(\mathbf{L}^\gamma\mathbf{L}\Theta[y])\mathbf{L}_3 \mapsto_{\mathrm{R1}} \mathbf{L}(\mathbf{L}^{\gamma-2}\mathbf{L}\Theta[y])\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_\Gamma \mathbf{L}^2(\mathbf{L}^\gamma\mathbf{L}\Theta[y])\mathbf{L}_3 \mapsto_{\mathrm{R1}} \mathbf{L}^2(\mathbf{L}^{\gamma-2}\mathbf{L}_3\Theta[y])\mathbf{L}_3$$

$$\mapsto_{\mathrm{R8}} \mathbf{L}(\mathbf{L}^{\gamma-2}\mathbf{L}_3\Theta[y])\mathbf{L}_3.$$

(Recall that, if $\gamma = 1$, the reduction sequences have to be adapted.)
- $\Delta_2$ cannot be the same as $\mathbf{L}a \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3, \mathbf{LL}_3\mathbf{L}_3$, or $\mathbf{L}(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$, since in none of these terms the left subterm is $\mathbf{L}$.
- $\Delta_2 \equiv \mathbf{L}(\mathbf{L}a) \equiv x(yy)$. Thus, $a \equiv x \equiv y \equiv \mathbf{L}$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R6}} \mathbf{L}(\mathbf{L}a)\mathbf{L}_3 \mapsto_{\mathrm{R1}} \mathbf{L}_3\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R1}} \mathbf{L}(\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R9}} \mathbf{L}_3\mathbf{L}_3.$$

- $\mathbf{L}(\mathbf{L}a) \not\equiv x(\mathbf{LL}yy)$ or $x(\mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y)$, because neither $\mathbf{L} \equiv \mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)$ nor $\mathbf{L} \equiv \mathbf{LL}y$ are possible.
- $\mathbf{L}(\mathbf{L}a) \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3, \mathbf{LL}_3\mathbf{L}_3$, or $\mathbf{L}(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$.

$\Delta_1 \equiv \mathbf{L}^2(\mathbf{LL}a)\mathbf{L}_3$. There are two subterms to be considered: $\mathbf{LL}a$ and $\mathbf{L}(\mathbf{LL}a)$.
- $\Delta_2 \equiv \mathbf{LL}a \equiv x\Gamma[y]$; therefore, $x \equiv \mathbf{LL}$ and $a \equiv \Gamma[y]$. This is a repetition of the generic case for $\Delta_1 \equiv \mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3$, $\Delta_2 \equiv \mathbf{L}a$ above leading here to a common reduct $\mathbf{L}(\mathbf{L}^{\gamma-1}\mathbf{L}_3\Theta[y])\mathbf{L}_3$.
- $\Delta_2 \equiv \mathbf{LL}a \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3$, or $\mathbf{LL}_3\mathbf{L}_3, \mathbf{L}(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$.
- $\mathbf{L}(\mathbf{LL}a) \equiv x(yy)$. Thus, $a \equiv y \equiv \mathbf{LL}$ and $x \equiv \mathbf{L}$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R6}} \mathbf{L}(\mathbf{LL}(\mathbf{LL}))\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R1}} \mathbf{L}(\mathbf{LL}(\mathbf{LL}))\mathbf{L}_3.$$

- $\mathbf{L}(\mathbf{LL}a) \not\equiv x(\mathbf{LL}yy)$ or $x(\mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y)$, because neither $\mathbf{LL} \equiv \mathbf{LL}y$, nor $\mathbf{LL} \equiv \mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)$.
- $\mathbf{L}(\mathbf{LL}a) \not\equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{LL}x)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3$ or $\mathbf{LL}_3\mathbf{L}_3, \mathbf{L}(\mathbf{LL}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$: The left-hand subterms are too long.

$\Delta_1 \equiv \mathbf{L}^2(\mathbf{L}^n\mathbf{L}_3 a)\mathbf{L}_3$: Again we are confronted with two critical subterms: $\mathbf{L}^n\mathbf{L}_3 a$ and $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a)$.

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 a \equiv x\Gamma[y]$. Again this is like the case $\Delta_1 \equiv \mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3$, $\Delta_2 \equiv \mathbf{L}a$, differing only in that the common reduct is $\mathbf{L}(\mathbf{L}^{n+2}\mathbf{L}_3\Theta[y])\mathbf{L}_3$.
- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 a \equiv \mathbf{L}^2(\mathbf{L}x)\mathbf{L}_3$ implies $a \equiv \mathbf{L}_3$, $x \equiv \mathbf{L}^{n-3}\mathbf{L}_3$, and $n \geqslant 3$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R}8} \mathbf{L}(\mathbf{L}^n\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}6} \mathbf{L}(\mathbf{L}^{n-1}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R}6} \mathbf{L}^2(\mathbf{L}^{n-1}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}8} \mathbf{L}(\mathbf{L}^{n-1}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3.$$

- $\Delta_2 \equiv \mathbf{L}^n\mathbf{L}_3 a \equiv \mathbf{L}^2(\mathbf{L}\mathbf{L}x)\mathbf{L}_3$ implies and $a \equiv \mathbf{L}_3$, $x \equiv \mathbf{L}$, and $n=2$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R}8} \mathbf{L}(\mathbf{L}^2\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}7} \mathbf{L}(\mathbf{L}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R}7} \mathbf{L}^2(\mathbf{L}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}8} \mathbf{L}(\mathbf{L}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3.$$

- $\mathbf{L}^n\mathbf{L}_3 a \not\equiv \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 x)\mathbf{L}_3$, because $\mathbf{L}^{n-2}\mathbf{L}_3 \not\equiv \mathbf{L}^m\mathbf{L}_3 x$.
- $\mathbf{L}^n\mathbf{L}_3 a \equiv \mathbf{L}\mathbf{L}_3\mathbf{L}_3$ entails $a \equiv \mathbf{L}_3$ and $n=1$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R}8} \mathbf{L}(\mathbf{L}\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}8} \mathbf{L}(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R}8} \mathbf{L}^2(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3 \mapsto_{\mathrm{R}8} \mathbf{L}(\mathbf{L}_3\mathbf{L}_3)\mathbf{L}_3.$$

- $\mathbf{L}^n\mathbf{L}_3 x \not\equiv \mathbf{L}(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$.
- $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a) \equiv x(yy)$. Thus, $a \equiv y \equiv \mathbf{L}^n\mathbf{L}_3$, and $x \equiv \mathbf{L}$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R}6} \mathbf{L}(\mathbf{L}^n\mathbf{L}_3(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 \mapsto_{\mathrm{R}1} \mathbf{L}\mathbf{L}(\mathbf{L}^n\mathbf{L}_3)\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R}1} \mathbf{L}(\mathbf{L}\mathbf{L}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3 \mapsto_{\mathrm{R}10} \mathbf{L}\mathbf{L}(\mathbf{L}^n\mathbf{L}_3))\mathbf{L}_3.$$

- $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a) \equiv x(\mathbf{L}\mathbf{L}yy)$; thus, $x \equiv \mathbf{L}$, $a \equiv \mathbf{L}$, and $n=0$. Then

$$\Delta_1 \overset{\Delta_1}{\mapsto}_{\mathrm{R}6} \mathbf{L}(\mathbf{L}_3\mathbf{L})\mathbf{L}_3,$$

$$\Delta_1 \overset{\Delta_2}{\mapsto}_{\mathrm{R}1} \mathbf{L}(\mathbf{L}^2\mathbf{L}\mathbf{L})\mathbf{L}_3 \mapsto_{\mathrm{R}1} \mathbf{L}(\mathbf{L}_3\mathbf{L})\mathbf{L}_3.$$

- $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3 a) \not\equiv x(\mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y)$. This equivalence cannot hold, because its consequences $\mathbf{L}^n\mathbf{L}_3 a \equiv \mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3 y)y$ and $\mathbf{L}^n\mathbf{L}_3 \equiv \mathbf{L}^{m+1}(\mathbf{L}^m\mathbf{L}_3)$ are absurd.
- $\mathbf{L}(\mathbf{L}^n\mathbf{L}_3)$ never matches any of $\mathbf{L}^2(\mathbf{L}a)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}\mathbf{L}a)\mathbf{L}_3, \mathbf{L}^2(\mathbf{L}^m\mathbf{L}_3 a)\mathbf{L}_3, \mathbf{L}\mathbf{L}_3\mathbf{L}_3$, or $\mathbf{L}(\mathbf{L}\mathbf{L}(\mathbf{L}^m\mathbf{L}_3))\mathbf{L}_3$. The global pattern of $\Delta_2$ is $\mathbf{L}(\dots)$, whereas the other terms are of form $\mathbf{L}(\dots)\mathbf{L}_3$. $\square$

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